LIMIT

(B.Sc.-II, Paper-III)

Group- A

(Real Analysis)

Topic: - Limit of a function of one variable and related theorems.

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Limit point of a set $D \subseteq \mathbb{R}$:→

Definition :→ Let $D \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. Then a is said to be a limit point of D if for any $\delta > 0$, the interval $(a-\delta, a+\delta)$ contains at least one point of D other than a, i.e.;

$$D \cap \{x \in \mathbb{R} : 0 < |x-a| < \delta\} \neq \phi$$

Example :→

① If $D = (0,1) \cup \{2\}$, then 2 is not a limit point of D.
The set of all limit points of D is the closed interval $[0,1]$.

②. Every point of an interval is its limit point.

③ If $D = \{x \in \mathbb{R} : 0 < |x| < 1\}$, then every point in the interval $[-1,1]$ is a limit point of D.

④ If $D = \{\frac{1}{n} : n \in \mathbb{N}\}$, then 0 is the only limit point of D.

⑤ If $D = \{\frac{n}{n+1} : n \in \mathbb{N}\}$, then 1 is the only limit point of D.

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Definition (i) :→ For $a \in \mathbb{R}$, an open interval of the form $(a-\delta, a+\delta)$ for some $\delta > 0$ is called a <u>neighbourhood</u> of a; it is also called a δ-neighbourhood of a.

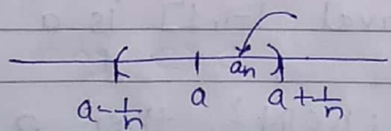
Definition (ii) :→ By a <u>deleted</u> <u>neighbourhood</u> of a point $a \in \mathbb{R}$ we mean a set of of the form $D_\delta := \{x \in \mathbb{R} : 0 < |x-a| < \delta\}$ for some $\delta > 0$ i.e., the set $(a-\delta, a+\delta) \setminus \{a\}$.

Remark :→ ① A point $a \in \mathbb{R}$ is a <u>limit point</u> of $D \subseteq \mathbb{R}$ if and only if every <u>deleted</u> neighbourhood of a contains at least one point of D.

THEOREM :→ A point $a \in \mathbb{R}$ is a <u>limit point</u> of $D \subseteq \mathbb{R}$ if and only if $\exists$ a sequence (a_n) in $D \setminus \{a\}$ such that $\lim\limits_{n\to\infty} a_n = a$.

Proof :→ (⇒) Suppose $a \in \mathbb{R}$ is a limit point of D.

We constract a sequence (a_n).
$\forall n \in \mathbb{N}$, $\exists\ a_n \in D \setminus \{a\}$ s.t. $a_n \in \left(a - \frac{1}{n}, a + \frac{1}{n}\right)$.



$\therefore \lim\limits_{n\to\infty} a_n = a$ proved.

(⇐) conversely, suppose that $\exists$ a sequence (a_n) in $D \setminus \{a\}$ s.t. $\lim\limits_{n\to\infty} a_n = a$.

Hence for any $\delta > 0$, $\exists$ a natural number $N \in \mathbb{N}$ s.t. $a_n \in (a-\delta, a+\delta)$ $\forall$ $n \geq N$.

$\therefore$ for $n \geq N \Rightarrow a_n \in (a-\delta, a+\delta) \cap D \setminus \{a\}$

$\therefore$ a is a limit point of D.

\# proved.

Limit of a function $f(x)$ as $x \to a$:→

Definition :→ Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be a real valued function, and let $a \in \mathbb{R}$ be a limit point of D. We say that $b \in \mathbb{R}$ is a limit point of $f(x)$ as ~~x→a~~ x approaches a if for every $\epsilon > 0$, $\exists$ $\delta > 0$ such that

$$|f(x) - b| < \epsilon \quad \text{whenever } x \in D \text{ and } 0 < |x-a| < \delta$$

And symbolically we write $\lim_{x \to a} f(x) = b$.

or $f(x) \to b$ as $x \to a$.

Example ① If $\lim_{x \to a} f(x)$ exists, prove that it must be unique.

Proof :→ Let, if possible $f(x)$ tends to limits l_1 and l_2 as $x \to a$.

Hence for any $\epsilon > 0$, it is possible to choose a $\delta > 0$, s.t.

$$|f(x) - l_1| < \frac{\epsilon}{2}, \quad \text{when } 0 < |x-a| < \delta$$

$$\& \ |f(x) - l_2| < \frac{\epsilon}{2}, \quad \text{whenever } 0 < |x-a| < \delta$$

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Now,

$$|l_1 - l_2| = |l_1 - f(x) + f(x) - l_2|$$
$$\leq |l_1 - f(x)| + |f(x) - l_2|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ , when } 0 < |x-a| < \delta$$

i,e; $|l_1 - l_2|$ is less than any positive number (however small) and so it must be equal to zero.

i,e; $|l_1 - l_2| = 0 \Rightarrow l_1 = l_2$

proved.

Example ② Let D be an interval and $a \in D$ or a is end point of D.

(i) Let $f(x) = x$. Since

$$|f(x) - a| = |x - a|, \forall x \in D$$

∴ For any $\epsilon > 0$.

$$|f(x) - a| < \epsilon \quad \text{whenever } 0 < |x-a| < \delta = \epsilon$$

Hence, $\lim_{x \to a} f(x) = a$

(ii) Let $f(x) = x^2$, P.T. $\lim_{x \to a} f(x) = a^2$

sol^n:→ For any $\epsilon > 0$

$$|f(x) - a^2| = (|x| + |a|)|x-a|, \forall x \in D, x \neq a$$

$$\because |x| = |x - a + a| \leq |x-a| + |a| \leq 1 + |a|$$

whenever $|x-a| < 1$,

We have

$$|f(x)-a^2| = (1+2|a|)\,|x-a|, \ \forall x \in D, \ 0<|x-a|<1.$$

Therefor,

$x \in D, \ 0<|x-a| \leq 1,$

$$(1+2|a|)\,|x-a| < \epsilon \Rightarrow |f(x)-a^2| < \epsilon$$

Thus,

$$x \in D, \ 0<|x-a|<\delta := \min\left\{1, \frac{\epsilon}{1+2|a|}\right\} \Rightarrow |f(x)-a^2|<\epsilon.$$

Hence, $\lim_{x\to a} f(x) = a^2$

proved.

THEOREM:→ If $\lim_{x\to a} f(x) = b$, then $\exists$ a deleted neighborhood D_δ of a such that

$$|f(x)| \leq M \ , \forall x \in D_\delta \cap D$$

Proof:→ Suppose that $\lim_{x\to a} f(x) = b$.

Then $\exists$ a deleted neighborhood D_δ of a s.t.

$$|f(x)-b| < 1 \ \forall x \in D \cap D_\delta.$$

Hence

$$|f(x)| = |f(x) - b + b|$$

$$\leq |f(x-b| + |b| \leq 1+|b| \ , \forall x \in D \cap D_\delta.$$

Thus, $|f(x)| \leq M = 1+|b|$, $\forall x \in D \cap D_\delta$.

proved.

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limit of a function in terms of sequences :→

THEOREM :→ Let a be a limit point of $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. ~~Then~~ If $\lim_{x\to a} f(x) = b$, then

for every sequence (x_n) in D s.t. $x_n \to a$

we have $f(x_n) \to b$.

Proof :→ Suppose that $\lim_{x\to a} f(x) = b$.

Let (x_n) be a sequence in D s.t. $x_n \to a$.

Let $\epsilon > 0$ be given. We have to show that $\exists\ n_0 \in \mathbb{N}$ s.t. $|f(x_n) - b| < \epsilon$, $\forall\ n \geq n_0$

$\because \lim_{x\to a} f(x) = b$, we know that $\exists\ \delta > 0$ s.t.

$$x \in D,\ 0 < |x-a| < \delta \Rightarrow |f(x) - b| < \epsilon \quad \text{—(1)}$$

Also since $x_n \to a$.

Therefor, $\exists$ $N \in \mathbb{N}$ s.t.

$|x_n - a| < \delta$, $\forall\ n \geq N$.

Hence from (1), we have

$$|f(x_n) - b| < \epsilon, \quad \forall\ n \geq N.$$

$\therefore \lim_{n\to\infty} f(x_n) = b$ proved.

THEOREM :→ If for every sequence (x_n) in D which converges to a, the sequence $(f(x_n))$ converges to b then $\lim_{x\to a} f(x) = b$.

Proof :→ Suppose that every sequence (x_n) in D which converges to a, the sequence $(f(x_n))$ converges to b.

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Sol. Assume that f does not have the limit b as $x \to a$.

Then, $\exists\ \epsilon_0 > 0$ s.t. for every $\delta > 0$, $\exists$ at least one $x_\delta \in D$ s.t.

$$0 < |x_\delta - a| < \delta \quad \text{and} \quad |f(x_\delta) - b| > \epsilon_0$$

In particular, for every $n \in \mathbb{N}$, $\exists\ x_n \in D$ s.t.

$$0 < |x_n - a| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - b| > \epsilon_0$$

Thus, $x_n \to a$ but $f(x_n) \not\to b$.

This is contradiction to our hypothesis.

Remark:→ Suppose (x_n) is a sequence in $D \setminus \{a\}$ s.t. $x_n \to a$.

① If $(f(x_n))$ does not converge, then $\lim_{x \to a} f(x)$ does not exists.

② If $(f(x_n))$ does not converge to a given $b \in \mathbb{R}$, then either $\lim_{x \to a} f(x)$ does not exists.

or $\lim_{x \to a} f(x)$ exists but $\lim_{x \to a} f(x) \neq b$.

③ If (y_n) is another sequence in $D \setminus \{a\}$ which converges to a and the sequences $(f(x_n))$ and $(f(y_n))$ converge to different points, then $\lim_{x \to a} f(x)$ does not exists.

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Example :→ Prove that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exists.

Proof :→

Let $f(x) = \sin\left(\frac{1}{x}\right)$.

Consider the sequence $\{x_n\} = \left\{\frac{1}{n\pi}\right\}$ and $\{y_n\} = \left\{\frac{1}{\frac{\pi}{2} + 2n\pi}\right\}$. Then both $x_n \to 0$ & $y_n \to 0$.

But $f(x_n) = \sin\left(\frac{1}{x_n}\right) = \sin(n\pi) = 0 \to 0$, while

$f(y_n) = \sin\left(\frac{1}{y_n}\right) = \sin\left(\frac{\pi}{2} + 2n\pi\right) = 1 \to 1$.

Thus $\lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(y_n)$.

$\therefore \lim_{x \to 0} f(x)$ does not exists.

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THEOREM:→ Suppose $\lim_{x\to a} f(x) = b$ and $\lim_{y\to b} g(y) = c$.

If D_1 and D_2 are the domains of f and g respectively, and if $f(x) \in D_2 \setminus \{b\}$ for every $x \in D_1 \setminus \{a\}$, then $\lim_{x\to a} g(f(x)) = c$.

Proof:→ Let $\epsilon > 0$ be given.

Then $\exists$ $\delta_1 > 0$ such that

$$0 < |y-b| < \delta_1 \Rightarrow |g(y) - b| < \epsilon$$

Also, let $\delta_2 > 0$ be such that

$$0 < |x-a| < \delta_2 \Rightarrow |f(x) - b| < \delta_1$$

Hence along with the given condition that $f(x) \in D_2 \setminus \{b\}$ for every $x \in D_1 \{a\}$,

$$0 < |x-a| < \delta_2 \Rightarrow 0 < |f(x) - b| < \delta_1 \Rightarrow |g(f(x)) - c| < \epsilon$$

$\therefore$ $\lim_{x\to a} g(f(x)) = c$ proved.

Example (1) If $f(x)$ is a polynomial, say $f(x) = a_0 + a_1 x + \cdots\cdots + a_k x^k$, then for any $a \in \mathbb{R}$

$$\lim_{x\to a} f(x) = f(a)$$

Proof:→ Let $b = f(a)$ and let $\epsilon > 0$ be given.

We have to find $\delta > 0$ s.t.

$|x-a|<\delta \Rightarrow |f(x)-b|<\epsilon$.

$\because f(x)-f(a) = (x-a)\left[x^{n-1}+x^{n-2}a+\cdots\cdots+xa^{n-2}+a^{n-1}\right]$

Now, suppose $|x-a|<1$. Then $|x|<1+|a|$ so that

$$|x^{n-j}a^{j-1}| < (1+|a|)^{n-1}$$

and hence,

$$|x^n-a^n| < |x-a|\cdot n(1+|a|)^{n-1}$$

Thus, $|x-a|<1$ implies

$$|f(x)-f(a)| \le |x-a|\left(|a_1| + |a_2|\cdot 2(1+|a|) + \cdots\cdots + |a_k|\, k\,(1+|a|)^{k-1}\right)$$

Therefor, taking $\alpha := |a_1| + |a_2|\, 2\,(1+|a_1|) + \cdots\cdots + |a_k|\, k\,(1+|a|)^{k-1}$,

We have

$$|f(x)-f(a)|<\epsilon, \text{ whenever } |x-a|<\delta := \min\{1, \tfrac{\epsilon}{\alpha}\}.$$

proved

LEFT limit and RIGHT Limit: →

Definition: → Let f be a real valued function defined on a set $D \subseteq \mathbb{R}$, and let $a \in \mathbb{R}$ be limit point of D.

(i) We say that $f(x)$ has the left limit $b \in \mathbb{R}$ as $x \to a$ if for every $\epsilon>0$, $\exists\ \delta>0$ s.t.

$$|f(x)-b|<\epsilon, \text{ whenever } x \in D,\ a-\delta<x<a$$

and in that case we write

$$\lim_{x \to a^-} f(x) = b \quad \text{or } f(x) \to b \text{ as } x \to a^-.$$

(ii) We say that $f(x)$ has the right limit $b \in \mathbb{R}$ as $x \to a$ if for every $\epsilon > 0$, $\exists\ \delta > 0$ s.t.

$$|f(x) - b| < \epsilon \quad \text{whenever } x \in D,\ a < x < a+\delta.$$

and in that case we write

$$\lim_{x \to a^+} f(x) = b \quad \text{or } f(x) \to b \text{ as } x \to a^+.$$

Remark:→ We have the following characterizations in terms of sequences

① $\lim_{x \to a^-} f(x) = b$ if and only if for every sequence (x_n) in $D \setminus \{a\}$

$x_n < a,\ \forall\ n \in \mathbb{N},\ x_n \to a \Rightarrow f(x_n) \to b.$

② $\lim_{x \to a^+} f(x) = b$ if and only if for every sequence (x_n) in $D \setminus \{a\}$,

$x_n > a,\ \forall\ n \in \mathbb{N},\ x_n \to a \Rightarrow f(x_n) \to b.$

(Proof is left as an excercise).

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Limit at ∞ and at $-\infty$:→

Definition:→ Suppose a function f is defined on an interval of the form (a, ∞) for some $a \in \mathbb{R}$. Then we say that $f(x)$ has the limit b as $x \to \infty$, if for every $\epsilon > 0$, $\exists\ M > a$ s.t.

$$|f(x) - b| < \epsilon \quad \text{whenever } x > M$$

and in that case we write

$$\lim_{x \to \infty} f(x) = b.$$

Definition:→ Suppose a function f is defined on an interval of the form $(-\infty, a)$ for some $a \in \mathbb{R}$. Then we say that $f(x)$ has the limit point b as $x \to -\infty$, if for every $\epsilon > 0$, $\exists\ M < a$ s.t.

$$|f(x) - b| < \epsilon \quad \text{whenever } x < M$$

and in that case we $\lim_{x \to -\infty} f(x) = b$.

Example:→ We show that $\lim_{x \to \infty} \frac{1}{x} = 0$.

Proof:→ Taking $f(x) = \frac{1}{x}$ for $x \neq 0$, $b = 0$ and $\epsilon > 0$, we observe that

$$|f(x) - b| < \epsilon \iff \frac{1}{|x|} < \epsilon \iff |x| > \frac{1}{\epsilon}$$

Hence,

$$x > \frac{1}{\epsilon} \Rightarrow |x| > \frac{1}{\epsilon} \Rightarrow |f(x) - b| < \epsilon$$

This ~~observe~~ shows that $|f(x) - b| < \epsilon$ whenever $x > M := \frac{1}{\epsilon}$.

$\therefore \lim_{x \to \infty} \frac{1}{x} = 0$ proved.

Example (ii) :→ Show that $\lim_{x \to -\infty} \frac{1}{x} = 0$.

Proof:→ Taking $f(x) = \frac{1}{x}$ for $x \neq 0$, $b = 0$ and $\epsilon > 0$, we observe that

$$|f(x) - b| < \epsilon \Leftrightarrow \frac{1}{|x|} < \epsilon \Leftrightarrow |x| > \frac{1}{\epsilon}$$

Hence,

$$x < -\frac{1}{\epsilon} \Rightarrow |x| > \frac{1}{\epsilon} \Rightarrow |f(x) - b| < \epsilon$$

This shows that

$|f(x) - b| < \epsilon$ whenever $x < M := -\frac{1}{\epsilon}$.

Example (iii) Show that $\lim_{x \to \infty} \frac{1}{x^2} = 0$

Proof:– Taking $f(x) = \frac{1}{x^2}$ for $x \neq 0$, $b = 0$ & $\epsilon > 0$

We observe that,

$$|f(x) - b| < \epsilon \Leftrightarrow \frac{1}{x^2} < \epsilon \Leftrightarrow |x| > \frac{1}{\sqrt{\epsilon}}$$

Hence,

$$x > \frac{1}{\sqrt{\epsilon}} \Rightarrow |x| > \frac{1}{\sqrt{\epsilon}} \Rightarrow |f(x) - b| < \epsilon$$

This shows that

$|f(x) - b| < \epsilon$ whenever $x > M := \frac{1}{\sqrt{\epsilon}}$

Example (ix) Show that $\lim_{x \to \infty} \frac{1+x}{1+x^2} = 0$

Proof:→

Let $f(x) = \frac{1+x}{1+x^2}$ for $x \in \mathbb{R}$.

$$f(x) = \frac{1+x}{1+x^2} = \frac{\frac{1}{x^2} + \frac{1}{x}}{\frac{1}{x^2} + 1} \to \frac{0}{1} = 0.$$

Definition: We define the following

(1) $\lim_{x \to a} f(x) = \infty$ if for ~~any~~ every $M > 0$,

$\exists\ \delta > 0$ such that

$$0 < |x-a| < \delta \Rightarrow f(x) > M$$

(2) $\lim_{x \to a} f(x) = -\infty$ if for ~~any~~ every $M > 0$,

$\exists\ \delta > 0$ such that

(3) $\lim_{x \to +\infty} f(x) = \infty$ if for any $M > 0$, $\exists\ \alpha > 0$ s.t.

$$x > \alpha \Rightarrow f(x) > M$$

(4) $\lim_{x \to +\infty} f(x) = -\infty$ if for ~~any~~ every $M > 0$,

$\exists\ \alpha > 0$ such that

$$x > \alpha \Rightarrow f(x) < -M$$

(5) $\lim_{x \to -\infty} f(x) = \infty$ if for every $M > 0$, $\exists\ \alpha > 0$ s.t.

$$f(x) < -\infty \Rightarrow f(x) > M$$

(6) $\lim_{x \to -\infty} f(x) = -\infty$ if for every $M > 0$, $\exists\ \alpha > 0$ s.t.

$$x < -\alpha \Rightarrow f(x) < -M$$

Remark:- It can be easily verify that

$$\lim_{x \to a} f(x) = \infty \Leftrightarrow \lim_{x \to a} [-f(x)] = -\infty$$

$$\lim_{x \to +\infty} f(x) = \infty \Leftrightarrow \lim_{x \to +\infty} [-f(x)] = -\infty$$

$$\lim_{x \to -\infty} f(x) = \infty \Leftrightarrow \lim_{x \to +\infty} [-f(x)] = -\infty$$

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Example(i) show that $\lim_{x \to 0} \frac{1}{x^2} = \infty$

Proof: $\rightarrow$ Let $f(x) = \frac{1}{x^2}$ for $x \neq 0$ & $M > 0$, ~~we~~

We obsurve that

$$f(x) > M \Leftrightarrow \frac{1}{x^2} > M \Leftrightarrow |x| < \frac{1}{\sqrt{M}}$$

Hence, for $0 < \delta < \frac{1}{\sqrt{M}}$

$$|x| < \delta \Rightarrow |x| < \frac{1}{\sqrt{M}} \Rightarrow f(x) > M$$

Thus $\lim_{x \to 0} \frac{1}{x^2} = 0$

Example(ii) We show that $\lim_{x \to 1} \left| \frac{1+x}{1-x} \right| = \infty$

Let $f(x) = \left| \frac{1-x}{1+x} \right|$ for $x \neq 1$.

Then for $M > 0$

$$f(x) = \left| \frac{1+x}{1-x} \right| > M \Leftrightarrow |1-x| < \frac{|1+x|}{M}$$

and

$$|1+x| = |2 - (1-x)| \geq 2 - |1-x| > 1 \text{ whenever } |x-1| < 1.$$

Hence,

$$|1-x| < 1 \ \& \ |x-1| < \frac{1}{M} \Rightarrow |1-x| < \frac{|1+x|}{M} \Rightarrow f(x) > M$$

Thus,

$$|x-1| < \delta := \min\{1, \tfrac{1}{M}\} \Rightarrow f(x) > M$$

shaving that $\lim_{x \to 1} \left| \frac{1+x}{1-x} \right| = \infty$.

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Example (iii) Let $f(x) = x^2$, $x \in \mathbb{R}$.

We show that $\lim\limits_{x \to \infty} f(x) = \infty$ & $\lim\limits_{x \to -\infty} f(x) = \infty$.

For $M > 0$

$$f(x) = x^2 > M \Leftrightarrow |x| > \sqrt{M}$$

$$\therefore x > \sqrt{M} \Rightarrow f(x) > M$$

$$x < -\sqrt{M} \Rightarrow f(x) > M.$$

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(Q1) Prove that the function

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases}$$

has a limit at x_0 if and only if $x_0 = 0$

Proof:→ We First ~~prove~~ prove that $\lim_{x \to 0} f(x) = 0$.

Let $\epsilon > 0$ & choose $\delta = \epsilon$.

Then $0 < |x - 0| < \delta \Rightarrow |f(x) - 0| = |f(x)| = |x| < \epsilon$

Next, show that if $x_0 \neq 0$, $\lim_{x \to x_0} f(x)$ does not exists.

$\exists$ sequences $\{r_n\}$ of rationals and $\{z_n\}$ of irrationals converging to x_0.

Then $f(r_n) = r_n \to x_0$.

Whereas $f(z_n) = -z_n \to -x_0$.

Since $x_0 \neq 0$, $\lim_{x \to x_0} f(r_n) \neq \lim_{x \to x_0} f(z_n)$

$\therefore \lim_{x \to x_0} f(x)$ does not exists.

Example (2) Use sequential criterion to prove that $\lim_{x \to x_0} f(x)$ does not exists.

Dirichlet's function : $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$;

$x_0 \in \mathbb{R}$.

Example :→ Prove that $\lim_{x \to 0} x \sin \frac{1}{x} = 0$

Solution :→ Let $\epsilon > 0$ be given. choose $\delta = \epsilon$.

Then if $0 < |x - 0| < \delta \Rightarrow 0 < |x| < \delta$, we have

$$|f(x) - 0| = \left|x \sin \tfrac{1}{x} - 0\right|$$

$$= \left|x \sin \tfrac{1}{x}\right|$$

$$= |x| \left|\sin \tfrac{1}{x}\right|$$

$$\Rightarrow |f(x) - 0| \leq |x| \qquad \left(\because \left|\sin \tfrac{1}{x}\right| \leq 1\right)$$

$$< \delta = \epsilon$$

Hence $\lim_{x \to 0} f(x) = 0$ proved.

LIMIT VS. ONE - SIDED LIMITS

THEOREM :→ If x_0 is a cluster point of $D(f) \cap (-\infty, x_0)$, and a cluster point of $D(f) \cap (x_0, \infty)$, then $\lim_{x \to x_0} f(x) = L \Leftrightarrow$ both

$\lim_{x \to x_0^-} f(x) = L$ and $\lim_{x \to x_0^+} f(x) = L$.

Proof :→ Suppose x_0 is a cluster point of $D(f) \cap (-\infty, x_0)$, and a cluster point of $D(f) \cap (x_0, \infty)$.

($\Rightarrow$) Suppose $\lim_{x \to x_0} f(x) = L$.

Let $\epsilon > 0$. Since $\lim_{x \to x_0} f(x) = L$,

$\therefore \exists \delta > 0$, s.t. $\forall x \in D(f)$, $0 < |x - x_0| < \delta$

$\Rightarrow |f(x) - L| < \epsilon$.

Then $\forall x \in D(f)$

$x_0 - \delta < x < x_0 \Rightarrow 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$; &

$x_0 < x < x_0 + \delta \Rightarrow 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$.

Therefor, $\lim_{x \to x_0^-} f(x) = L$ & $\lim_{x \to x_0^+} f(x) = L$.

($\Leftarrow$) Suppose that $\lim_{x \to x_0^-} f(x) = L$ and $\lim_{x \to x_0^+} f(x) = L$.

Let $\epsilon > 0$.

$\because \lim_{x \to x_0^-} f(x) = L$, $\exists \delta_1 > 0$ s.t. $\forall x \in D(f)$,

$x_0 - \delta_1 < x < x_0 \Rightarrow |f(x) - L| < \epsilon$

Since $\lim_{x \to x_0^+} f(x) = L$, $\exists \delta_2 > 0$ s.t. $\forall x \in D(f)$,

$x_0 < x < x_0 + \delta_2 \Rightarrow |f(x) - L| < \epsilon$

choose $\delta = \min\{\delta_1, \delta_2\}$.

Then, $\forall x \in D(f)$, $0 < |x - x_0| < \delta \Rightarrow$ either

$x_0 - \delta < x < x_0$ or $x_0 < x < x_0 + \delta$.

In either of these cases,

$|f(x) - L| < \epsilon$.

Thus.

$\forall x \in D(f)$, $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$.

$\therefore \lim_{x \to x_0} f(x) = L$ proved.

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THANK YOU